Compressed Sensing Matrices: Binary vs. Ternary

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Abstract

This paper evaluates the behavior of $\{0, 1\}$ binary matrix and $\{0, \pm 1\}$ ternary matrix in compressed sensing. Supposing that they share the same distribution on nonzero positions, binary matrix shows favorable performance over ternary matrix with the observation on RIP analysis and numerical simulations.

Index Terms

binary matrix, ternary matrix, compressed sensing, RIP.

I. Introduction

Recently, a variety of random [1], [2] or deterministic matrices [3], [4], [5] with the form of binary or ternary, are successively proposed in compressed sensing for their low complexity and comparable performance with Gaussian matrix. However, in practical applications, one probably raises a question that has been ignored, that is, binary matrix vs. ternary matrix, if they hold the same distribution on nonzero positions, which one is better? This paper is developed to address this interesting problem.

Compressed sensing [6] attempts to extract useful information from sparse signal with a simple undersampling process

$$y = Ax \tag{1}$$

where $x \in \mathbb{R}^n$ denotes a $k(\ll n)$-sparse signal, $A \subset \mathbb{R}^{m\times n}$ with $m \ll n$ represents a sensing matrix, and $y \in \mathbb{R}^m$ is the observation. Restricted Isometry Property (RIP) of order k ensures that the k-sparse signal x could be uniquely recovered, through solving a l_1-minimization based linear programming problem

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$$\min ||\hat{x}||_1 \qquad \text{subject to} \quad y = A\hat{x} \tag{2}$$

Explicitly, RIP of order k defines the smallest constant δ_k with upper bound, e.g., equal to 0.307 in [7], while satisfying the following constraint

$$(1-\delta_k)||x_T||^2 \ll ||A_T x_T||^2 \ll (1+\delta_k)||x_T||^2 \tag{3}$$

for all subsets $T \subset \{1, ..., N\}$ with cardinality $|T| \leq k$ and all vectors $x_T \in \mathbb{R}^{|T|}$, where A_T denotes the set of columns of A with indices in T. In this sense, it can be observed that smaller δ_k indicates better sensing matrix of tolerance to larger sparsity k.

It is well known that the solution to δ_k could be transformed to seek the extreme singular values [8] of A_T, since

$$1-\delta_k \leq \lambda_{min} \leq \frac{||A_T x_T||^2}{||x_T||^2} \leq \lambda_{max} \leq 1+\delta_k \tag{4}$$

where λ_{min} and λ_{max} represent the two extreme eigenvalues of $A_T' A_T$. Note that, during the solution process to extreme eigenvalues, $A_T' A_T$ tends to be approximated as a symmetric matrix rather than a positive semidefinite matrix, because the relative accurate solution to the former is available for practical matrices [4]. In addition, it is worth recalling that RIP is more concerned with the minimal eigenvalue λ_{min} rather than the maximal eigenvalue λ_{max} [9]. So with $1-\delta_k \leq \lambda_{min}$, it appears that larger λ_{min} implies smaller δ_k, namely better sensing matrix.

The rest of this literature is organized as follows. In next Section, binary matrix and ternary matrix are defined first, then their RIP is analyzed. In Section III, numerical simulations are conducted for comparing the performance of above two types of matrices. Finally, we end this paper with a conclusion in Section IV.

II. Binary Matrix vs. Ternary Matrix

This section aims at comparing the performance between $\{0, 1\}$ binary matrix and $\{0, \pm 1\}$ ternary matrix with same distribution on nonzero entries. Precisely, binary matrix is customarily assumed to hold d nonzero entries in each column, $1 \leq d < m$, and the corresponding ternary matrix is generated by rendering the nonzero entries of binary matrix *bipolar* with *equal* probability.

To compute the extreme eigenvalues, two types of matrices are further characterized by the correlation between distinct columns. Assume that the correlation values between columns vary within the integer

interval $[0,\ s]$ for binary matrix, where $0 \leq s < d$, then it is easy to deduce that the correlation values for ternary matrix range within $[-s,\ s]$, since two types of matrices share the same nonzero distribution. Inversely, if ternary matrix is first given, the correlation values of binary matrix can also be derived accordingly. According to correlation distributions of above two types of matrices, we roughly derive their extreme eigenvalues in the following Corollaries 1 and 2, respectively.

By comparing Corollaries 1 and 2, we can observe that both types of matrices share the same λ_{max}, while binary matrix holds larger λ_{min}. As stated in the introduction, this larger λ_{min} indicates that binary matrix holds smaller δ_k, namely better performance. However, it should be noted that the extreme eigenvalues are achieved on the assumption that $A_T' A_T$ holds the element values s and $-s$ with some special distributions [10]. However, it seems hard to ensure that these distributions could be satisfied well for various practical matrices [4]. Therefore, strictly speaking, with the comparison between Corollaries 1 and 2, we can reasonably conjecture rather than assert, that binary matrix should outperform ternary matrix in compressed sensing.

Corollary 1 For symmetric matrix $A_T' A_T \subset \mathbb{R}^{k \times k}$ with diagonal elements equal to d and off-diagonal elements varying in $[0, s]$, its two extreme eigenvalues can be approximated as:

$$\lambda_{min} \geq \begin{cases} d - ks/2 & \text{if } k \text{ is even} \\ d - s\sqrt{k^2-1}/2 & \text{if } k \text{ is odd} \end{cases} \tag{5}$$

and

$$\lambda_{max} \leq d + (k-1)s \tag{6}$$

Proof: Refer to Theorem 1 and Corollary 2 in [10] ■

Corollary 2 For symmetric matrix $A_T' A_T \subset \mathbb{R}^{k \times k}$ with diagonal elements equal to d and off-diagonal elements varying in $[-s, s]$, its two extreme eigenvalues can be approximated as:

$$\lambda_{min} \geq d - (k-1)s \tag{7}$$

and

$$\lambda_{max} \leq d + (k-1)s \tag{8}$$

Proof: Refer to Theorem 1 and Corollary 2 in [10] ■

III. Numerical Simulations

A. Setup

This section empirically assesses two representative binary matrices: traditional *random* binary matrix [1] with size $(50, 200)$ and degree 5, and *deterministic* near-optimal binary matrix [4], with size $(200, 400)$ and degree 7. And their corresponding ternary versions are randomly produced in each simulation. Note that, as it is known, the decoding performance is sensitive to some distributions of sparse signal, especially when nonzero entries share the same magnitude [11], [12]. Thus two types of challenging sparse signals, separately from ternary set $\{0, \pm 1\}$ and binary set $\{0, 1\}$, are tested herein. Simultaneously, for generality, two types of sparse signals of nonzero entries separately drawn from $N(0, 1)$ and the positive part of $N(0, 1)$, are also evaluated. As for decoding algorithms, to derive a relative stable and convincing decoding performance, near-optimal decoding algorithm (Basis Pursuit) is exploited to recover the sparse signal with diverse sparsity levels. And the correct decoding precision is measured with $1 - ||\hat{x} - x||_2/||x||_2$. Each simulation point is derived after 1000 iterations.

B. Results

The decoding performance of random binary matrix [1] and deterministic near-optimal binary matrix [4] via input signals of diverse distributions and sparsity levels, are shown in Figures 1-4. With Figures 1 and 2, it is clear that the performance curves of above two types of matrices agree with each other. Precisely, binary version and ternary version shows comparable performance, as input sparse signal is of the distribution $\{0, \pm 1\}$; in contrast, binary version significantly outperforms ternary version, as sparse signal holds the distribution $\{0, 1\}$. In fact, we also get the similar result, if two types of sparse signals are separately drawn from $N(0, 1)$ and positive part of $N(0, 1)$, as shown in Figures 3 and 4. Explicitly, binary matrix dominates ternary matrix as input signals drawn from the positive part of $N(0, 1)$; otherwise, their performance difference is negligible. *Overall*, these results disclose that binary matrix has obvious advantage on sparse signal of same nonzero sign. Therefore, in terms of the overall performance and complexity, it can be argued that binary matrix is more competitive in practical applications.

IV. Conclusion

This paper has shown the interest for the behavior of binary matrix and ternary matrix in compressed sensing. As expected by RIP, binary matrix presents better overall performance than ternary matrix. This will encourage the practical application of binary matrix, especially when input signals consist of nonzero

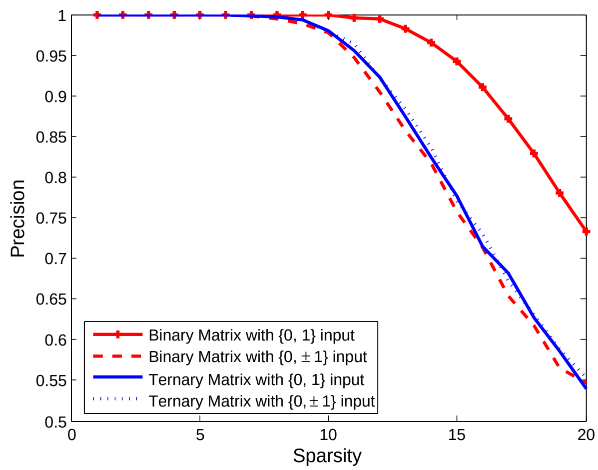

Fig. 1. The performance of traditional random binary matrix and its ternary version, over two types of input signals of distributions $\{0, 1\}$ and $\{0, \pm 1\}$.

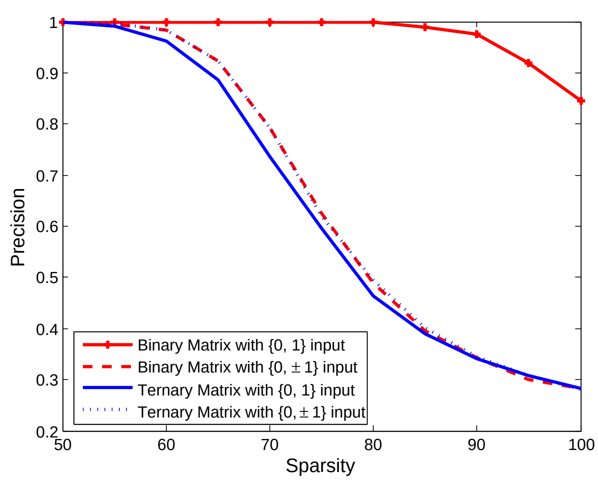

Fig. 2. The performance of deterministic near-optimal binary matrix and its ternary version, over two types of input signals of distributions $\{0, 1\}$ and $\{0, \pm 1\}$.

entries of the same sign. In future, it might be interesting to further interpret the robustness of binary matrix on sparse signal of same nonzero sign.

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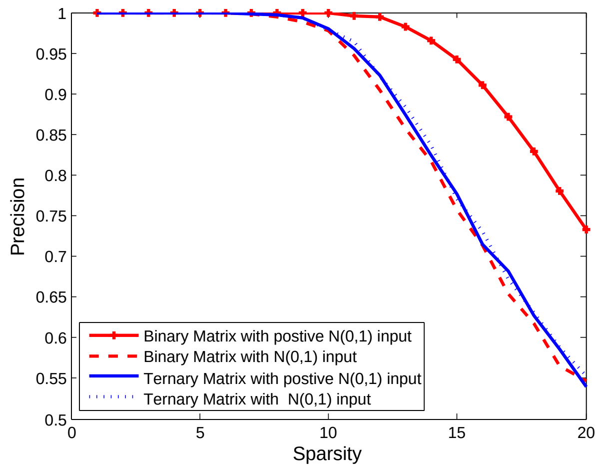

Fig. 3. The performance of traditional random binary matrix and its ternary version, over two types of input signals of distributions $N(0,1)$ and the positive part of $N(0,1)$.

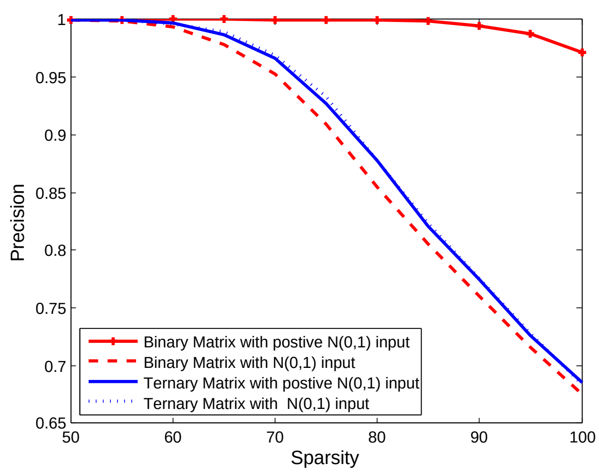

Fig. 4. The performance of deterministic near-optimal binary matrix and its ternary version, over two types of input signals of distributions $N(0,1)$ and the positive part of $N(0,1)$.

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